What is SEM?

– A new approach to mortality prediction –

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Abstract

Survival Energy Model (SEM) is a new approach to the mortality prediction introduced by Shimizu *et al.* [17]. The approach can give a cohort-wise distribution function of the time of death, which is defined as the first hitting time of a "survival energy" diffusion process to zero. In this paper, we propose a new SEM with some improvements of the prediction procedure given in [17]. Moreover, we shall discuss some practical advantages of SEM in practice.

Key words: Survival Energy Models; cohort-wise mortality prediction; diffusion processes, inverse Gaussian processes, SEM project.

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1 Introduction

The last several decades, statistical evidence show that the life time of human being is getting longer and longer in many countries. For example, the expected lifetime in Japan has been increasing, and the one in 2020 is 85 years old although 60 years old in 1950. Such a rapid change of longevity is called the *"Longevity Revolution"*, e.g., [22]. On one hand, this trend makes the human life selective and valuable for individuals. On the other hand, we are facing on many problems related to medical, economical, and social welfare situation. For example, the problem of financial collapse of the national pension has been a critical issue in Japan. In such a situation, the mortality prediction is becoming an important worldwide social issue.

Since the early 20th century, the mortality prediction has been studied by many authors, and a certain methodology seems to be already established in practice. In most of mortality models, the "death" is regarded the first event of a time-inhomogeneous Poisson process: let T_x be the remaining lifetime of an individual of age x, it is assumed that

$$\mathbb{P}(T_x > t+1 \,|\, T_x > t) = \exp\left(-\int_t^{t+1} \mu(x,s)\,\mathrm{d}s\right),$$

where $\mu(x,t)$ is a (possibly stochastic) intensity function or called the *force of mortality* in insurance context. The existing studies are going to find a model for $\mu(x,t)$. For example, some deterministic mortality models such as the Gompertz, the Makeham, or the Heligman-Pollard law were presented in earlier years; see, e.g., Olivieri [14], and many stochastic mortality models are recently presented; e.g., Biffis [1], Cairns *et al.* [5], Hainaut and

Devolder [10], Biffis *et al.* [2], Blackburn and Sherris [3] and the references therein. Moreover, by assuming that $\mu(x,\cdot)$ is constant between $(t,t+1]$, say $m(x,t)$, they are going to model the mortality $m(x,t)$ itself, and any established classical models correspond to such a type; the Lee-Carter model [8], the Renshaw-Haberman model [16], the CBD model [6, 7], among others. We call these approaches *reduced-form approach* because it considers the death is just a stochastic event.

However, Shimizu *et al.* [17] proposed a *structural approach* under the "Survival Energy Hypothesis", which is an assumption that believes the existence of the survival energy for human beings, and the death occurs if the energy vanishes. Shimizu *et al.* [17] used some inhomogeneous diffusion (ID) processes as the survival energy model (SEM) cohort-wisely, say $X^c = (X^c_t)_{t\geq 0}$ with the cohort c, which is called *ID-SEM*:

$$X^c_t = x_c + \int_0^t U_c(s)\,\mathrm{d}s + \int_0^t V_c(s)\,\mathrm{d}W_s, \tag{1.1}$$

where x_c is a positive constant corresponding to the initial survival energy, U_c and V_c are deterministic functions on $\mathbb{R}_+$ and W is a Wiener process. They define the time of death by the first hitting time for X^c to zero: $\tau^c := \inf\{t > 0 \,|\, X^c_t < 0\}$. They consider some parametric models for U and V, and illustrated that the *mortality function*:

$$q_c(t) := \mathbb{P}(\tau^c \leq t) \tag{1.2}$$

can fit very much to the empirical distribution function of the time of death. This indicates that the SEM has a hight potential to propose a good parametric family to predict the future's mortality functions nevertheless it is just a fictitious assumption.

In this paper, we propose a new SEM with a procedure to improve the prediction. Moreover, we shall discuss some advantages of the SEM compared to the classical regression type models in the reduced-form approach. Finally, we shall introduce the SEM project, which gives the cohort/country-wise mortality functions explicitly with the values of the parameters in a web site.

2 A general procedure of mortality prediction via SEM

Let $X^c = (X_t)_{t\geq 0}$ be an underlying survival energy process of cohort c, and $q_c(\cdot)$ given in (1.2) be the mortality function of that cohort. In the SEM approach, we consider a specific parametric model $X^{c,\theta}$ with a parameter $\theta \in \Theta$, where Θ is a parameter space that is open and bounded in $\mathbb{R}^p$. Then we have a parametric model for the mortality function:

$$q_c(t,\theta) = \mathbb{P}(\tau^c_\theta \leq t),$$

where $\tau^c_\theta := \inf\{t > 0 \,|\, X^{c,\theta}_t \leq 0\}$. We assume that, for each cohort c, there exists the true value of the parameter, say θ_c, such that

$$q_c(\cdot,\theta_c) \equiv q_c(\cdot).$$

2.1 Parameter estimation

For the each SEM, the unknown parameter θ_c is estimated by the least-squares fitting between the explicit (conditional) mortality function for a suitably chosen threshold $S > 0$:

$$q_c(t|S) := \mathbb{P}(\tau^c \leq t \,|\, \tau^c > S) = \frac{q_c(t)}{1 - q_c(S)}.$$

Note that the above $q_c(\cdot|S)$ should be actually estimated by the individual data τ_i^c, which is the time of death of i-th individual in the cohort c, based on national statistics:

$$\widehat{q}_c(t|S) := \frac{\displaystyle\sum_{i=1}^{n_c} \mathbf{1}_{\{S<\tau_i^c\le t\}}}{\displaystyle\sum_{i=1}^{n_c} \mathbf{1}_{\{\tau_i^c>S\}}}, \tag{2.1}$$

where n_c is the population in the cohort c, and this converges to the true $q_c(t|S)$ uniformly in t when $n_c \to \infty$. However, such individual data are not available in practice. However, a version (or an approximation) can be produced from the **Human Mortality Database (HMD)** [20] as follows.

Data Processing from HMD. (R-code for this transform is available at [21]) Let $q_x^{(c)}$ be the death probability within 1 year of age x in 'calender year' c, which are directly found in HMD [20]. Moreover, let q_x^c be the mortality rate in 1 year of age x with 'birth year' c (cohort).
In the life table of the 'calender year' c, it holds that

$$q_0^{(c)} = q_0^c, \quad q_1^{(c)} = q_1^{c-1}, \quad \dots, \quad q_\omega^{(c)} = q_\omega^{c-\omega},$$

where ω is the final age of the life table ($\omega = 110$ in HMD).

(1) From those, we have that

$$q_k^c = q_k^{(c+k)}, \quad k = 2, 3, \dots, \omega.$$

(2) Compute the survival probability $p_k^c := 1 - q_k^c$ for $k = 0, 1, \dots, \omega$. Then

$$\mathbb{P}(\tau^c > t|\tau^c > S) = p_S^c \cdot p_{S+1}^c \cdots p_{t-1}^c$$

for $t = S+1,\ S+2,\ \dots,\ \omega$.

(3) As a consequence, we have

$$\widehat{q}_c(t|S) = 1 - \prod_{k=S}^{t-1} p_k^c,$$

for $t = S+1,\ S+2,\ \dots,\ \omega$.

Definition 2.1. For a given $S > 0$ with $q_c(S) > 0$ and $t_d > \cdots > t_2 > t_1 > S$, the (conditional) LSE for θ_c is given by

$$\widehat{\theta}_c := \arg\min_{\theta\in\overline{\Theta}} \sum_{i=1}^{d} \left|q_c(t_i, \theta|S) - \widehat{q}_c(t_i|S)\right|^2.$$

The consistency and the asymptotic normality of the LSE are shown in [17]. We shall recall them (the statements in [17] have some typos).

Theorem 2.1. Suppose that $q_c(t,\cdot|S) \in C^1(\Theta)$ for each $t > 0$ and $S > 0$. Moreover, for a given $S > 0$, suppose the following identifiability condition holds true:

$$q_c(t_i,\theta|S) = q_c(t_i,\theta'|S) \text{ for } i = 1,2,\ldots,d \quad \Rightarrow \quad \theta = \theta'. \tag{2.2}$$

Then the LSE $\widehat{\theta}_c$ has the weak consistency as the sample size of the corresponding cohort n_c in (2.1) increases:

$$\widehat{\theta}_c \xrightarrow{\mathbb{P}} \theta_c, \quad n_c \to \infty.$$

Theorem 2.2. Suppose the same assumptions as in Theorem 2.1. Moreover, suppose $q_c \in C^2(\Theta)$ and Θ is a convex subset of $\mathbb{R}^m$. Then,

$$\sqrt{n_c}(\widehat{\theta}_c - \theta_c) \xrightarrow{\mathcal{D}} R_d^{-1} Q_d \cdot N_d(0,\Sigma), \quad n_c \to \infty$$

where

$$Q_d = (\partial_\theta q_c(t_1,\theta_c|S),\ldots,\partial_\theta q_c(t_d,\theta_c|S) \in \mathbb{R}^m \otimes \mathbb{R}^d,$$
$$R_d = \sum_{i=1}^{d} \left[(\partial_\theta q_c)(\partial_\theta^\top q_c)(t_i,\theta_c|S) + \{q_c(t_i,\theta) - q_c(t_i,\theta_c)\}\partial_\theta^2 q_c(t_i,\theta_c|S)\right] \in \mathbb{R}^m \otimes \mathbb{R}^m,$$

and the variance-covariance matrix $\Sigma = (\sigma_{ij})_{1\le i,j\le m}$ is given by

$$\sigma_{ij} = \frac{1}{\overline{q}_c^2(S)}\Lambda(t_i,t_j) + \frac{\overline{q}_c(t_i)\overline{q}_c(t_j)}{\overline{q}_c^4(S)}\Lambda(S,S) - \frac{\overline{q}_c(t_i)}{\overline{q}_c^3(S)}\Lambda(t_j,S) - \frac{\overline{q}_c(t_j)}{\overline{q}_c^3(S)}\Lambda(t_i,S)$$

with $\Lambda(x,y) = q_c(x \wedge y) - q_c(x)q_c(y)$ and $\overline{q}_c = 1 - q_c$.

2.2 Prediction of mortality functions with a modification

Suppose that we have some estimated values of θ_c for some cohorts $c_1 < c_2 < \cdots < c_m$, say $\widehat{\theta}_{c_1}, \widehat{\theta}_{c_2}, \ldots, \widehat{\theta}_{c_m}$. We suppose that the future's parameter θ_c is determined in the following form:

$$\theta_c = h(c) + \epsilon_c, \quad \epsilon_c \sim N_p(0,\sigma_\epsilon^2), \tag{2.3}$$

for a deterministic (unknown) mean function h. Assuming that the estimated parameter $\widehat{\theta}_{c_1}, \widehat{\theta}_{c_2}, \ldots, \widehat{\theta}_{c_m}$ are realizations of θ_{c_i} $(i = 1,\ldots,m)$, we estimate h that is suitably parametrized. Once h is estimated, say $\widehat{h}$, we will predict a parameter $\theta_{c'}$ for a future's cohort c' by

$$\widehat{\theta}_{c'} = \widehat{h}(c'), \quad c' > c_m, \tag{2.4}$$

and we can obtain the predictive mortality function $q_{c'}(\cdot,\widehat{\theta}_{c'})$. In particular, the α-prediction interval for $\theta_{c'}$ is also obtained by

$$\widehat{I}_\alpha^{c',m} := \left[\widehat{\theta}_{c'} - z_{\alpha/2}\widehat{\sigma}_\epsilon,\ \widehat{\theta}_{c'} + z_{\alpha/2}\widehat{\sigma}_\epsilon\right], \tag{2.5}$$

where $\widehat{\Sigma}$ is an estimator of Σ in (2.3), and z_α is $(1-\alpha)$-percentile of $N(0,1)$. That is, it follows that

$$\lim_{m\to\infty} \mathbb{P}\left(\theta_{c'} \in \widehat{I}_\alpha^{c',m}\right) = \alpha$$

Remark 2.1 (Modified mortality function)**.** Although Shimizu *et al.* [17] use $q_{c'}(\cdot, \widehat{\theta}_{c'})$ itself for the predicted mortality function, we shall adjust parameters within the α-prediction interval so that the mortality function can fit the existing data for the cohort c'. That is, when the empirical data $\widehat{q}_{c'}(t|S)$ for $t = t_1, \dots, t_{d'}$ already exists, we reselect the predictor so that

$$\widetilde{\theta}_{c'} = \arg \min_{\theta \in \widehat{I}_\alpha^{c',m}} \sum_{i=1}^{d'} |q_{c'}(t_i, \theta|S) - \widehat{q}_{c'}(t_i|S)^2, \tag{2.6}$$

where $\widehat{I}_\alpha^{c',m}$ is given in (2.5). We shall use $q_{c'}(\cdot, \widetilde{\theta}_{c'})$ as a predicted mortality function, which can often improve the prediction. We call it the *modified predicted mortality function (MPMF)*. Later, we shall compare the direct prediction (2.4) with the above modification (2.6) in some examples.

3 SEM with an explicit mortality function

3.1 ID-SEM

The original SEM is given as a time-inhomogeneous diffusion process (1.1), which is called ID-SEM in Shimizu *et al.* [17], and it has a restriction such that

$$\frac{U_c(t)}{V_c^2(t)} = \frac{\kappa_c}{2} \in \mathbb{R}, \tag{3.1}$$

with $\inf_{t>0} V_c^2(t) > 0$. Shimizu *et al.* [17] propose the following parametric models for U_c and V_c:

$$\begin{aligned} U_c(t, \theta_c) &= \alpha_c + \beta_c \exp\left(\gamma_c (t - T_c)\right) \mathbf{1}_{\{t \geq T_c\}}; \\ V_c(t, \theta_c) &= \sqrt{\frac{2}{\kappa_c} U_c(t, \theta_c)}, \end{aligned}$$

where the parameter space Θ of $\theta_c = (\alpha_c, \beta_c, \gamma_c, \kappa_c)$ is given by

$$\Theta \subset \{(\alpha, \beta, \gamma, \kappa) \in \mathbb{R}^4 \,|\, \alpha < 0,\, \beta < 0,\, \gamma > 0,\, \kappa < 0\}.$$

Thanks to (3.1), the mortality function has a closed expression; see Molini *et al.* [**?**].

Theorem 3.1. The mortality function of ID-SEM is given by

$$q_c^{ID}(t, \theta_c) = 1 - \int_0^\infty f(z, t|\theta_c)\, \mathrm{d}z, \quad t \geq 0,$$

where $f(z, t|\theta_c) = G_{\theta_c}(z - x_c, t) - e^{-\kappa_c x_c} G_{\theta_c}(z + x_c, t)$;

$$G_{\theta_c}(y, t) := \frac{1}{2\sqrt{\pi S(t, \theta_c)}} \exp\left(-\frac{(y - M(t, \theta_c))^2}{4S(t, \theta_c)}\right);$$

$$M(t, \theta_c) = \int_0^t U(s, \theta_c)\, \mathrm{d}s; \quad S(t, \theta_c) = \frac{1}{2} \int_0^t V^2(s, \theta_c)\, \mathrm{d}s.$$

3.2 A new SEM: IG-SEM

We say that a random variable Y follows an *inverse Gaussian distribution*:

$$Y \sim IG(a, b),$$

with mean a and variance a^3/b if the probability density is given by

$$f_Y(y; a, b) = \sqrt{\frac{b}{2\pi y^3}} \exp\left(-\frac{b(y-a)^2}{2a^2 y}\right), \quad y > 0.$$

Definition 3.1 (IG-SEM; Inverse Gaussian)**.** We say X^c follows *IG-SEM* if

$$X_t^c = x_c - Y_t^c, \quad t \geq 0, \tag{3.2}$$

where $x_c > 0$ is an initial energy, $Y^c \sim IG(\Lambda_c, \sigma_c)$ is an *inverse Gaussian process* with mean function Λ_c and a parameter $\sigma_c > 0$, that is, $Y_0^c = 0$ *a.s.* and Y^c has independent increments. Moreover, for any $t > s > 0$ and an increasing function Λ_c with $\Lambda_c(0) = 0$, it follows that

$$Y_t^c - Y_s^c \sim IG\left(\Lambda_c(t) - \Lambda_c(s), \sigma_c(\Lambda_c(t) - \Lambda_c(s))^2\right).$$

Remark 3.1. Note that if $\Lambda(t) = t$, then Y is the inverse Gaussian Lévy process, which is a spectrally positive pure-jump subordinator. Hence IG-SEM has a jump in the path of survival energy although the path of ID-SEM is continuous.

Such a process is used to modeling the time of system failure in engineering, where the failure occurs at τ^c if the accumulating damages Y_t^c exceeds a certain threshold x_c: $\tau^c = \inf\{t > 0 \,|\, Y_t^c > x_c\}$, which is the same idea as our SE for human death; see Ye and Chen [19]. Then the mortality function is given by the following theorem.

Theorem 3.2. The mortality function for IG-SEM is given by

$$q_c^{IG}(t, \theta_c) = \Phi\left(\sqrt{\frac{\sigma_c}{x_c}}(\Lambda_{\theta_c}(t) - x_c)\right) - e^{2\sigma_c \Lambda_{\theta_c}(t)} \Phi\left(-\sqrt{\frac{\sigma_c}{x_c}}(\Lambda_{\theta_c}(t) + x_c)\right),$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, \mathrm{d}z$.

Later, we shall take the mean function Λ_{θ_c} as

$$\Lambda_{\theta_c}(t) = e^{a_c t} + b_c t - 1, \quad \theta_c = (a_c, b_c, \sigma_c) \in \Theta,$$

where

$$\Theta \subset \{(a, b, \sigma) \in \mathbb{R}^3 \,|\, a > 0,\ b > 0,\ \sigma > 0\}.$$

4 Advantages of SEM

4.1 Actuarial notation, computation and estimation

In actuarial mathematics, there are many complicate and unique notations only for actuaries. For example, the "single (net) premium of m-payment n-year terminable annuity" is written as $\ddot{a}^{(m)}_{x:\overline{n}|}$, and the "sigle (net) premium of n-year term immediate insurance" is written as $\bar{A}^1_{x:\overline{n}|}$ in the classical actuarial notation. Such a notation called the *Halo notation*, which was accepted at the 2nd International Congress of Actuaries (ICA1898), seems hard for non-actuarial people to understand, and it seems to make hard to approach to the actuarial

mathematics for beginners of this field even for mathematicians or statisticians. However, rewriting them in terms of a mortality function, which we say *SEM notation*, one can easily understand the meaning of them. For example, the simple notation such as ${}_tp_x$ is usually explained in a sentence that "the probability that an individual of age x survives for t years", but it may be not so clear for mathematician. If we write it in terms of the mortality function such that

$$ {}_tp_x := \frac{1-q_c(x+t)}{1-q_c(x)} = \mathbb{P}(\tau^c > x+t | \tau^c > x), $$

then we can say it is a "conditional survival function at age x", and the meaning will be much clearer for mathematicians and statisticians. Moreover, the usual researchers may not be familiar with the term "forth of mortality" of age x at time t, which is usually written as μ_{x+t}. However, it would be better for statisticians to say the "conditional hazard function of age x":

$$ \mu_{x+t} := -\frac{\mathrm{d}}{\mathrm{d}t} \log \mathbb{P}(\tau^c > x+t | \tau^c > x) \ \left(= \frac{\partial_t q_c(x+t)}{1-q_c(x+t)} \right) $$

Although there are many complicate "Halo notation", it would be convenient for non-actuarial users to rewrite them in terms of the mortality function. Then, anybody can easily compute actuarial quantities cohort-wisely once the explicit form of q_c is given: e.g.,

$$ \ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1}{m} \sum_{s=0}^{mn-1} v^{\frac{s}{m}} \frac{1-q_c\left(x+\frac{s}{m}\right)}{1-q_c(x)}; \qquad \bar{A}^1_{x:\overline{n}|} = \int_0^n v^t \frac{\partial_t q_c(x+t)}{1-q_c(x)} \, \mathrm{d}t, $$

where v is a discount factor and $\partial_\theta q_c$ is the derivative of the mortality function q_c, among others. We shall give many examples of those transformations in Appendix.

The largest advantages for expression in terms of the mortality functions are in statistical estimation for actuarial quantities. Consider the single premium of the whole life insurance at age x, say A_x, it is written in both ways as follows:

$$ \begin{aligned} A_x &:= \sum_{k=1}^{\infty} v^k \, {}_{k-1|}q_{x+k-1} \quad \text{(Halo notation)} \\ &= \sum_{k=1}^{\infty} v^k \frac{q_c(x+k) - q_c(x+k-1)}{1-q_c(x)} \quad \text{(SEM notation)} \end{aligned} $$

where $v \in (0,1)$ is a discount factor. If we use the Lee-Carter model, then it is written as

$$ A_x = \sum_{k=1}^{\infty} v^k \left[1 - \exp\left(-m_{x+k-1,t}(\alpha_{x+k}, \beta_{x+k}) \right) \right], $$

where $m_{x,t}$ is the (crude) mortality parametrized by

$$ m_{x,t}(\alpha_x, \beta_x) = \exp\left(\alpha_x + \beta_x \kappa_t + \epsilon_{x,t} \right), $$

with parameters α_x, β_x, which are to be estimated and κ_t, which is to be predicted usually by a time series model including some unknown parameters, and $\epsilon_{x,t}$, which is a noise process. In this case, we have to estimate many parameters $\{(\alpha_y, \beta_y)\}_{y=x,x+1,\dots}$ and those in κ_t, which can make the statistical error for A_x increase. However, if we use SEM, then the cohort-wise computation

$$ A_x = \sum_{k=1}^{\infty} v^k \frac{q_c(x+k, \theta_c) - q_c(x+k-1, \theta_c)}{1-q_c(x, \theta_c)} $$

requires only one parameter estimation for θ_c because θ_c is independent of $k = 1, 2, \dots$, which can make the statistical error less than the classical mortality models.

4.2 Sensitivity analysis

As is already seen in the previous section, most of the actuarial quantities are written in the functionals of the mortality function $q_c(t, \theta_c)$ with a few unknown parameters θ_c. This situation is good for *sensitivity analysis* with respect to the parameter change.

Consider an actuarial quantity for age x and cohort c represented by a Stieljes-type integral form such as

$$H(\theta) := \int_0^\infty Q_{c,x}(t|\theta)\,\mathrm{d}h(t), \quad \theta \in \Theta$$

where h is a function on $[0, \infty)$, and

$$Q_{c,x}(t|\theta) := \frac{q_c(x+t,\theta)}{1-q_c(x,\theta)},$$

and the meaning of integral sign is that $\int_0^\infty := \int_{[0,\infty)}$. We shall suppose the exchangeability of $\int_0^\infty$ and the differentiation ∂_θ up to we need:

$$\partial_\theta H(\theta) = \int_0^\infty \partial_\theta Q_{c,x}(t|\theta)\,\mathrm{d}h(t) < \infty, \quad \theta \in \Theta,$$

and this is continuous in θ.

Note that most of actuarial quantities are written in this form; see Tables **??**–**??**. For example, A_x, the single premium of the whole life insurance at age x is given by taking

$$h(t) = \sum_{k=1}^\infty v^k \left(\mathbf{1}_{\{t\geq k\}} - \mathbf{1}_{\{t\geq k-1\}}\right), \quad t \geq 0,$$

where $v \in (0,1)$. Moreover, for its immediate payment version

$$\bar{A}_x = \int_0^\infty v^t \frac{\partial_t q_c(x+t)}{1-q_c(x)}\,\mathrm{d}t$$

is given by $H(\theta)$ with

$$h(t) = -v^t - \mathbf{1}_{\{t\geq 0\}}.$$

Indeed, it follows by the integration-by-parts that

$$\begin{aligned} H(\theta) &= \int_0^\infty \frac{q_c(x+t,\theta)}{1-q_c(x,\theta)}\,(-v^t)'\mathrm{d}t - \frac{q_c(x,\theta)}{1-q_c(x,\theta)} \\ &= \left[-v^t \frac{q_c(x+t,\theta)}{1-q_c(x,\theta)}\right]_{t=0}^\infty + \int_0^\infty v^t \frac{\partial_t q_c(x+t,\theta)}{1-q_c(x,\theta)}\,\mathrm{d}t - \frac{q_c(x,\theta)}{1-q_c(x,\theta)} \\ &= \int_0^\infty v^t \frac{\partial_t q_c(x+t,\theta)}{1-q_c(x,\theta)}\,\mathrm{d}t = \bar{A}_x. \end{aligned}$$

We are interested in a difference $H(\theta) - H(\theta_0)$ for different values of parameters θ and θ_0. By Taylor's formula,

$$H(\theta) - H(\theta_0) = \int_0^\infty \partial_\theta Q_{c,x}(t|\theta_0)\,\mathrm{d}h(t) \cdot (\theta - \theta_0) + o(\theta - \theta_0).$$

The integral $\int_0^\infty \partial_\theta Q_{c,x}(t|\theta_0)\,\mathrm{d}h(t)$ can be evaluated by direct computation.

For our LSE $\widehat{\theta}_c$ of θ_0 given in Theorem 2.2 and the sample size n_c to obtain the estimator, we have by the delta method in statistics that

$$\sqrt{n_c}\left(H(\widehat{\theta}_c) - H(\theta_c)\right) = \int_0^\infty \partial_\theta Q_{c,x}(t|\theta_c)\,\mathrm{d}h(t) \cdot \sqrt{n_c}(\widehat{\theta}_c - \theta_c) + o_p(1)$$
$$\to^d N_p(0, \Sigma_{c,x}), \quad n_c \to \infty,$$

where the asymptotic variance $\Sigma_{c,x}$ is estimable by estimators of R_d, Q_d, Σ in Theorem 2.2 and the plug-in estimator $\displaystyle\int_0^\infty \partial_\theta Q_{c,x}(t|\widehat{\theta}_c)\,\mathrm{d}h(t)$. This can yield a confidence interval of $H(\theta_c)$:

$$\mathbb{P}\left(H(\theta_c) \in \left[H(\widehat{\theta}_c) - z_{\alpha/2}\frac{\widehat{\Sigma}_{x,c}}{\sqrt{n_c}},\ H(\widehat{\theta}_c) + z_{\alpha/2}\frac{\widehat{\Sigma}_{x,c}}{\sqrt{n_c}}\right]\right) \approx 1 - \alpha,$$

where z_α is the upper α-percentile of the standard normal distribution and $\widehat{\Sigma}_{c,x}$ is an estimator of the asymptotic variance $\Sigma_{c,x}$.

5 Concluding remarks: SEM project

We have the **SEM project**, successive researches after Shimizu *et al.* [17], which is a plan to make lists of cohort-wise (modified) mortality functions of each country, using ID- and IG-SEMs (or others if a better model is found). The mortality functions are predicted using data from HMD [20]. Thanks to those, anyone can use mortality functions of required cohort immediately without any computation, and we hope that it will be an alternative mortality database after HMD [20].

Visiting the website

`https://www.shimizu.sci.waseda.ac.jp/smzlab/semproject/`.

one can already find the Japanese mortality functions (male and female) in each cohort with values of parameters. We hope that many practitioners and researchers can enjoy those functions to compute cohort-wise premiums and liabilities of insurance, measuring longevity risks, and demographic researches.

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